Classifying fusion categories and subfactors

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What is a fusion category?

- A fusion category is a $\otimes$-category with duals, which is semisimple and has finitely many simple objects.
- Every modular tensor category gives a fusion category: just forget the braiding.
- The Drinfeld centre of a fusion category is a modular tensor category.

Why study fusion categories?

- A fusion category is precisely the algebraic data encoding a local topological field theory in $1+1$ dimensions.
- We can construct exotic examples, whose centres are exotic modular tensor categories.
- We have good techniques to classify them.

Principal graphs

Choose 'your favourite object' X in a fusion category $\mathcal{C}$.

Definition

The principal graph for $(\mathcal{C}, X)$ *has vertices for the simple objects, and an edge between* Y *and* Z *for each copy of* Z *appearing inside* $Y \otimes X$.

If the fusion category is unitary, the graph norm (=largest eigenvalue of the adjacency matrix) is equal to the categorical dimension of X.

Hereafter, we'll assume all our categories are unitary.

What's out there?

The dimensions of fusion objects are highly constrained. Jones proved the first result in this direction.

Theorem (Jones, Index for subfactors, '83)

If $1 < d(X) < 2$, *then* $d(X) = 2\cos(\pi/n)$.

The subfactor literature implicitly began the study of fusion categories a decade earlier: you should think of a (finite depth, II_1) subfactor as a pair of fusion categories along with a Morita equivalence between them.

Highly ahistorically, I'll give an arithmetic proof:

Proof.

The dimension is the largest eigenvalue of the principal graph. The only real algebraic integers less than 2 which are maximal amongst their conjugates are the numbers $2\cos(\pi/n)$. □

Theorem (Coste-Gannon, '94)

The dimension of an object in a fusion category is a cyclotomic integer.

Proof.

Entries of the S-matrix of the Drinfeld center are cyclotomic. In fact, the Galois group acts faithfully on the S-matrix and can be described as either row permutations or column permutations. These commute, so the Galois group itself is abelian. □

Using these results, we can get purely arithmetic constraints on possible dimensions of objects in fusion categories.

Theorem (Calegari-Morrison-Snyder, CMP '10)

Let X be a fusion object with dimension between 2 and $73/33 = 2.303030\ldots$. Then $d(X)$ is equal to one of the following algebraic integers:

$$\begin{aligned}
\frac{\sqrt{7}+\sqrt{3}}{2} &= 2.188901059\ldots \\
\sqrt{5} &= 2.236067977\ldots \\
1+2\cos(2\pi/7) &= 2.246979603\ldots \\
\frac{1+\sqrt{5}}{\sqrt{2}} &= 2.288245611\ldots \\
\frac{1+\sqrt{13}}{2} &= 2.302775637\ldots
\end{aligned}$$

Proof.

There are constraints between the maximum norm of a Galois conjugate of a cyclotomic integer, and the number of roots of unity we need to write it as a sum. Exploiting these, we reduce the possible cyclotomic integers which are maximal amongst their conjugates to finitely many cases. □

In fact, each of these dimensions is realized by an example.

But what's out there? Can we completely classify fusion categories with an object with small dimension?

Theorem (Ocneanu, Izumi, Kawahigashi, Nadal, 80s and 90s)

If a fusion category $\mathcal{C}$ is $\otimes$-generated by a self-dual object with dimension between 1 and 2, then $\mathcal{C}$ is an 'ADE category', that is

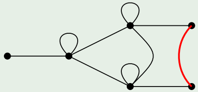
- *isomorphic to $\mathcal{C}(\mathfrak{sl}_2, \ell)$ for some ℓ (type A),*
- *a two-fold quotient of one of these (type D),*
- *or one of two sporadic cases, E_6 and E_8.*

- These categories include the physically relevant ones in the topological phases seen (or suspected) in fractional quantum Hall liquids.
- We can probably drop the hypothesis that the generator is self-dual.
- There's a similar classification of fusion categories generated by an object of dimension 2.

Enumerating principal graphs

We also remember which objects are dual to each other.

Example (The even part of the Haagerup subfactor)



The principal graph must satisfy an associativity test:

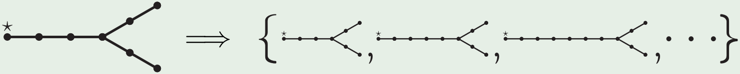
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such graphs with norm below some number L up to any rank or depth, obtaining a collection of allowed vines and weeds.

Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

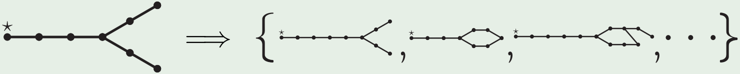
Example

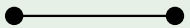


Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

Example

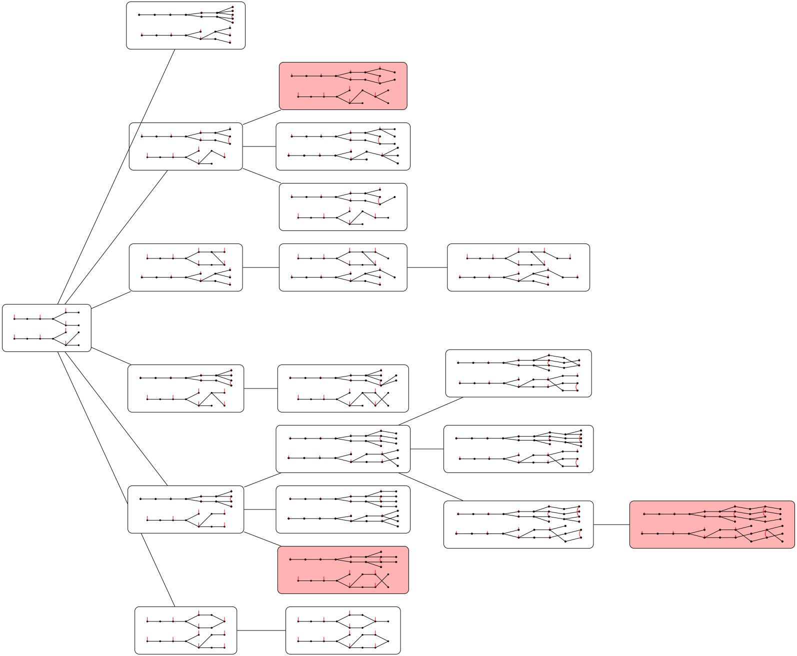


The weed  trivially represents all possible principal graphs (of singly generated fusion categories).

We can always convert a weed into a vine, at the expense of finding all possible depth 1 extensions of the weed (which stay below the norm limit, and satisfying the associativity condition) and adding these as new weeds.

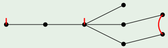
The is a finite problem, since high valence implies large graph norm, and graph norm increases under inclusions.

If the weeds run out, we go home happy. Realistically, we stop with some surviving weeds, and have to rule these out 'by hand'.



Theorem (Izumi-Jones-Morrison-Penneys-Peters-Snyder-Tener)

There is only one unitary fusion category $\otimes$-generated by a self-dual object with dimension in the interval $(2, \sqrt{5})$, namely the 'Izumi-Ostrik-Xu' category, with principal graph



Actually, we proved in Subfactors with index less than 5, parts 1, 2, 3 and 4 the corresponding result for subfactors, of which this is an easy consequence.

Theorem (in preparation)

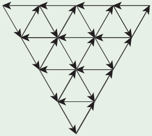
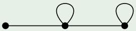
There are either one, two or three unitary fusion categories $\otimes$-generated by a self-dual object with dimension $\sqrt{5}$.

- As usual, we prove the subfactor result first; there are exactly 5 subfactors with index 5.
- Of these, 2 obviously can't be realised as fusion categories, one is a Tambara-Yamagami category, and I haven't yet checked the other 2 cases yet.

We're now working above $d = \sqrt{5}$, and have some partial results.

Conjecture (Morrison-Peters, last week)

For any unitary fusion category $\mathcal{C}$ $\otimes$-generated by an object X with dimension in the interval $(\sqrt{5}, \sqrt{3+\sqrt{5}})$, either

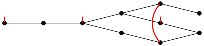
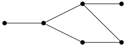
1. *The principal graph is*  , *in which case $\mathcal{C}$ must be isomorphic to $SU(3)_4$, or*
2. *the principal graph is* , *in which case $\mathcal{C}$ is isomorphic to a certain category constructed by Wenzl, or*
3. $X \otimes X^* \cong \mathbf{1} \oplus Z$, *for some simple object Z.*

(Case 1 is actually contained in case 3.)

In case 3 we can give many further constraints, but they are hard to summarize and unsatisfactory. I strongly suspect that there are no examples hiding in case 3.

'Proof'.

In fact, we classify 1-supertransitive subfactors. (This corresponds to ignoring case 3.) Using the same techniques as in our series on subfactors below index 5, we reduce to certain infinite families. The vines are all eliminated by number theoretic methods (in these infinite families, only finitely many cases can have cyclotomic norm), except for cases 1 and 2. For the weeds, we consider the tetrahedral $(6-j)$ symbols with two opposite edges labelled by the object X (this is the theory of 'connections'), and prove inequalities contradicting unitarity. □

The subfactors corresponding to cases 1 and 2 have principal graphs  and  .

Just as the exceptional Lie groups were discovered via the Killing-Cartan program of classification, the classification of small index subfactors is producing examples of exotic subfactors.

The new exotic subfactors and fusion categories we are finding

- constrain possible structural theorems,
- give counterexamples to conjectures, and
- give interesting new examples of modular data and thus exotic 3-manifold invariants.

Exotic fusion categories

A classical theorem of Brauer shows that the representation theory of any finite group can be defined over a cyclotomic field. (The same holds for quantum groups at roots of unity.) Etingof, Nikshych and Ostrik asked if this is true of every fusion category.

Theorem (Morrison-Snyder, Transactions of the AMS '10)

The even parts of the Haagerup and extend Haagerup subfactors cannot be defined over any cyclotomic field.

Proof.

Using the skein theory, we produce a canonical element of the ground field which is not cyclotomic. □