

## Subfactors with index in the range $(5, 3 + \sqrt{5})$

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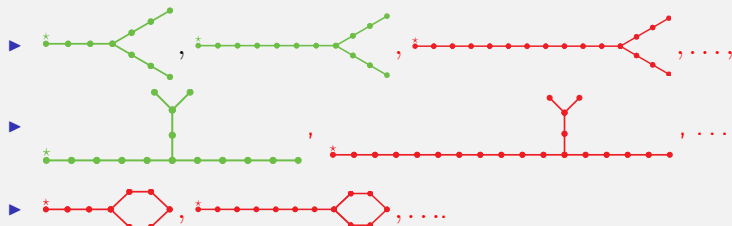
## Some history

Over the past twenty years, we've been progressively improving the classification of small index subfactors.

- ▶ ADE classification of subfactors with index less than 4, by Ocneanu, then Izumi, Kawahigashi, and Nadal.
- ▶ classification of subfactors with index 4, in terms of affine Dynkin diagrams and finite extra data, by Popa.

## Haagerup's list

- ▶ In 1993 Haagerup classified possible principal graphs for subfactors with index in  $(4, 3 + \sqrt{3})$ :



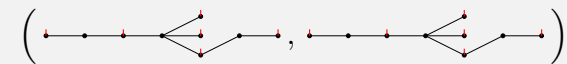
- ▶ Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- ▶ Haagerup (unpublished), Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- ▶ Two years ago we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. [arXiv:0909.4099](https://arxiv.org/abs/0909.4099)

In Subfactors with index less than 5, parts I, III, IIII and IV, we completed the classification up to index 5.

### Theorem (Izumi-Jones-Morrison-Penneys-Peters-Snyder-Tener)

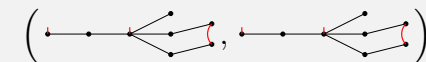
The only subfactors with index in  $[3 + \sqrt{3}, 5)$  have

- ▶ principal graph  $A_\infty$ , or
- ▶ are the 3311 GHJ subfactor (or its dual),



with index  $3 + \sqrt{3}$ , or

- ▶ are the 2221 Izumi-Xu subfactor (or its conjugate),



with index  $\frac{5 + \sqrt{21}}{2}$ .

# Today

Today I'm going to give a progress report on the next steps.

This is very recent work, and these slides are the first time I've written down the results.

Some of it might be wrong!

## Proof (continued).

- ▶ If  $n = 2$ 
  - ▶ if  $X_0 = 1$  then  $\dim X < 2$ , so we were below index 4,
  - ▶ if  $X_0 \geq \sqrt{2}$ , then the index is at least  $3 + \sqrt{5}$ .
- ▶ If  $n \geq 3$ , the index is greater than 5.04....

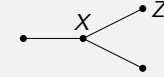
□

## Theorem (Theorem 5.1 in Part II)

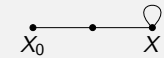
There are no 1-supertransitive subfactors with index in  $(4, 5)$ .

### Proof.

- ▶ There's no room for a quadruple point or double edge.



- ▶ An object  $Z$  at depth 2 has  $\dim Z < 2$ , and generates an ADET fusion category.
- ▶ Since  $X \in X \otimes Z$ , the principal graph for  $- \otimes Z$  must contain a  $T_n$ , with  $X$  the last vertex.
- ▶ If  $X_0$  is the first vertex in the  $T_n$ , we have  $\dim X = [n]_{q=\zeta_{4n+2}} \dim X_0$ .



(continued...)

## Proposition

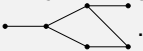
The only finite-depth 1-supertransitive subfactor with index in  $(4, 3 + \sqrt{5})$  has principal graph

### Proof.

- ▶ An even object with dimension in  $(2, \sqrt{5})$ , must have dimension  $\frac{\sqrt{3} + \sqrt{7}}{2}$ , by the classification of subfactors with index less than 5.
- ▶ If this holds for all objects at depth 2,  $\dim X^2 \geq 1 + \sqrt{3} + \sqrt{7} > 3 + \sqrt{5}$ .
- ▶ Thus an even object  $Z$  at depth 2 has  $\dim Z < 2$ , and generates an ADET fusion category.

(continued...)

Proof (continued).

- ▶ Since  $X \in X \otimes Z$ , the principal graph for  $- \otimes Z$  must contain a  $T_n$ , with  $X$  the last vertex.
- ▶ If  $X_0$  is the first vertex in the  $T_n$ , we have  $\dim X / \dim X_0 = [n]_{q=\zeta_{4n+2}}$ .
- ▶ If  $n = 2$ 
  - ▶ if  $X_0 = 1$  then  $\dim X < 2$ , so we were below index 4,
  - ▶ if  $X_0 \geq \sqrt{2}$ , then the index is at least  $3 + \sqrt{5}$ .
- ▶ If  $n = 3$ ,  $\dim X$  is the largest root of  $\lambda^3 - 2\lambda^2 - \lambda + 1$ , and fairly readily the entire principal graph is determined to be
 
- ▶ If  $n \geq 4$ , the index is greater than 8.29....

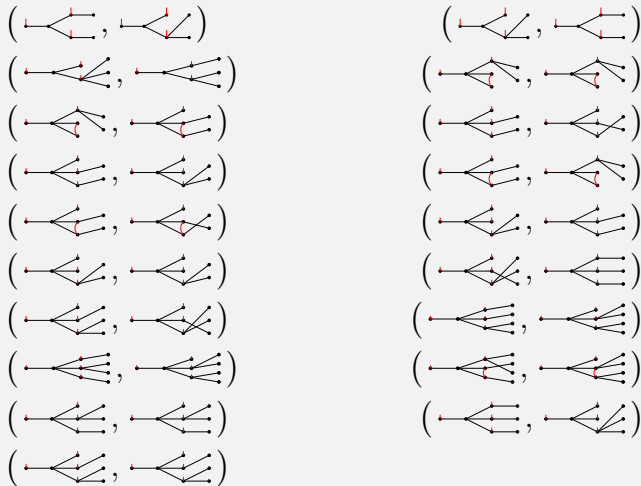
□

What happens if we drop the finite depth condition?

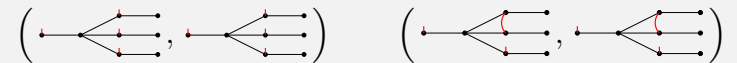
The above argument fails — an object at depth 2 could have dimensions in the interval  $(2, \sqrt{5})$ , generating an infinite depth  $A_\infty$  subcategory.

Happily, we can simply run the odometer, as in Part II.

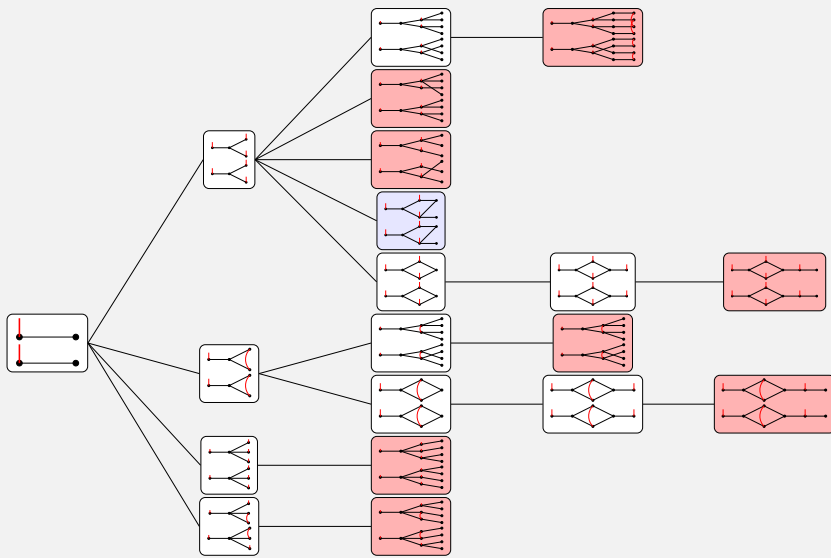
We can ignore the following graphs, because they must have an intermediate subfactor, which isn't possible with index in  $(5, 3 + \sqrt{5})$ .

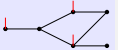


We can ignore the graphs



via the argument above; at least one of the three objects at depth 2 has dimension  $< 2$ .



- ▶ The graph  has a depth 2 object with dimension less than 2, and we handle this special case as above.
- ▶ For each red pair, some object has an impossible dimension.

Putting this together, we have

### Theorem (Morrison-Peters)

The only 1-supertransitive subfactors with index in the range  $(5, 3 + \sqrt{5})$  either

- ▶ have principal graph  $A_\infty$ , or
- ▶ have principal graph  $(\text{graph 1}, \text{graph 2})$ .

### Theorem (Wenzl)

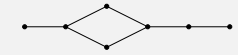
There is a unique subfactor with this principal graph.

His construction was via the BMW algebra: the subfactor is braided and related to  $SU(3)$ . It also appears in Evans & Pugh's recent work.

Later, I'll show you a brute force approach to showing existence and uniqueness.

### Example

The hardest case is for the graph



The last two objects have dimensions

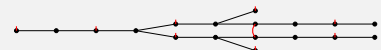
$$d_1 = \frac{1}{2}(q^4 - q^2 - 2 - q^{-2} + q^{-4})$$

$$d_2 = \frac{1}{2}(q^5 - q^3 - 3q - 3q^{-1} - q^{-3} + q^{-5}).$$

Now  $d_2 < 1.145$ , and  $d_2 = 1$  only if  $q = 1.69068...^{\pm 1}$ . But then  $d_1 = 1.54231...$ , which is not of the form  $2 \cos(\pi/n)$ .

What about higher supertransitivites in this range?

Recall that

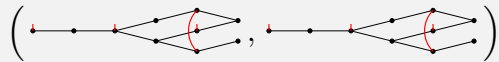
- ▶ the triple point obstruction exploits a certain inequality on the norms of certain connection entries, and
- ▶ to rule out the weed  in Part III, we found a certain identity between dimensions of objects, using the renormalization condition for a connection.

This suggests a new trick.

- ▶ Even when we only know the principal graph up to some depth, we can automate checking certain inequalities on the norms of a connection.
- ▶ Sometimes, this tells us that graph is impossible, or that we only need to consider that graph up to index  $L < 3 + \sqrt{5}$ .

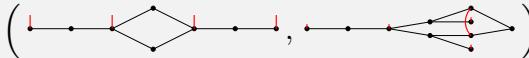
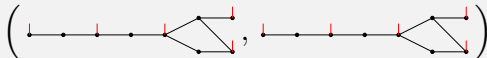
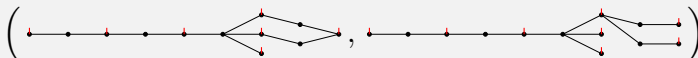
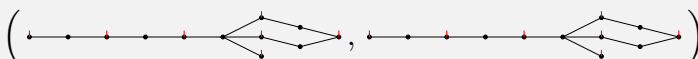
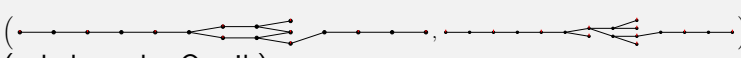
Even using this, the odometer still gets bogged down before reaching  $3 + \sqrt{5}$ .

The only possible graph we've found is



which is realized by the alternating part of  $SU(3)_4$ .  
 (And moreover is the unique subfactor with this principal graph.)  
 I already have some bets made with members of the audience that there's nothing besides these.

At index  $3 + \sqrt{5}$  and above, computer searches have thrown up some interesting possibilities.

- ▶  (see later in this talk)
- ▶  (see later)
- ▶  (ruled out by arguments in Part IIII)
- ▶  (open?)
- ▶  (ruled out by Ostrik)

and a few more ...

## Connections

With Emily Peters, I've written a computer program that finds bi-unitary connections.

- ▶ If the principal graph has no quadruple points, the program appears to 'always work'.
- ▶ It's fast: just a few seconds on all the examples we've tried, including extended Haagerup.

Finding bi-unitary connections isn't enough though. We need to determine the corresponding subfactor.

## Theorem

A bi-unitary connection  $K$  on  $\Gamma$  picks out a 'flat subalgebra'  $\mathcal{F}_K$  of the graph planar algebra  $\mathcal{G}(\Gamma)$ .

- ▶ The flat subalgebra might just be Temperley-Lieb.
- ▶ There's a condition on a connection, called flatness, ensuring

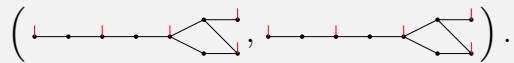
$$\Gamma(\mathcal{F}_K) = \Gamma$$

i.e. the principal graph of the flat subalgebra is the original principal graph.

Unfortunately, flatness specifies infinitely many conditions.

Our approach is the 'hybrid method'. We look for flat low weight vectors inside the graph planar algebra  $\mathcal{G}(\Gamma)$ .

- ▶ If there are no flat low weight vectors, there's no subfactor. This rules out



- ▶ Any flat low weight vector ensures that the flat subalgebra is strictly larger than Temperley-Lieb.
- ▶ If the first flat low weight vector lies in  $\mathcal{G}(\Gamma)_{2k}$ , then the corresponding subfactor is  $(k - 1)$ -supertransitive.

Sometimes (if we have classified the possible graphs), even this is enough to identify the principal graph.

## Example

Our program takes

- ▶ a few seconds to prove the existence of  $(\text{graph 1}, \text{graph 2})$ ,
- ▶ a few minutes to prove the existence of the Haagerup subfactor,
- ▶ a few hours to prove the existence of the Asaeda-Haagerup subfactor.

Those times might sound like a long time, but

- ▶ we're writing a much faster second version, and
- ▶ this was completely automatic!

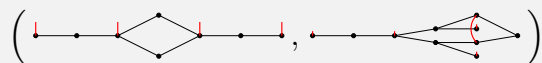
*Science is what we understand well enough to explain to a computer. Art is everything else we do.*

(Donald Knuth in the foreword to  $A = B$ .)

What can we do when we don't have any classification statements?

## Example

- ▶ There's a unique bi-unitary connection on



(at index  $3 + \sqrt{5}$ ).

- ▶ There's a flat low weight vector in  $\mathcal{G}(\Gamma)_6$ .
- ▶  $\therefore$  there's some 2-supertransitive subfactor at index  $3 + \sqrt{5}$ .

## One approach

We can compute all the projections in the flat subalgebra.

Given two idempotents  $P, Q \in \mathcal{G}(\Gamma)_{2k}$  we can compute

- ▶  $\text{Hom}_{\mathcal{G}(\Gamma)}(P, Q) = \{T \in \mathcal{G}(\Gamma)_{2k} \mid TP = QT = T\}$ , and
- ▶  $\text{Hom}_{\mathcal{F}(K)}(P, Q) = \text{Hom}_{\mathcal{G}(\Gamma)}(P, Q) \cap \mathcal{F}(K)_{2k}$ .

From this we can read off the principal graph of the flat subalgebra.

## Question

*Can we implement this efficiently enough for examples?*